# Optimization of Cascade Blade Mistuning, Part I: Equations of Motion and Basic Inherent Properties

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**The present work is divided into two main parts. Part I deals with the derivation of the equations of motion of mistuned compressor blades, where the aerodynamic coefficients are interpolated using quadratic expressions in the reduced frequency. It is shown that if the coefficients of the quadratic expressions are permitted to assume complex values, excellent accuracy is obtained, so that Padé rational expressions are not required (thus avoiding augmented states). Based on the resulting equations, it is analytically shown that the sum of all the real parts of the eigenvalues is independent of the mistuning introduced into the system. The problem of blade mistuning is further tackled using the aerodynamic energy approach, and the limiting vibration modes associated with purely alternate mistunings are identified. In Part II the aerodynamic energy approach is used to identify analytically the optimum mistunings together with their associated vibration modes. Numerical optimization of mistuning is employed and the existence of multiple optimum mistuning vectors is shown to exist. Part II also shows that any bending-torsion compressor blade system can be reduced to an equivalent purely torsional system of blades.**

## Introduction

THE introduction of inertia and/or mass mistunings into a tuned compressor blade system is known to consist of two main effects[1-6]: 1) mistuning always leads to an increase in flutter speed, and 2) mistuning may adversely affect the forced response of the system, thus leading to larger vibration levels and to larger stresses.

The present two-part work concentrates on gaining some physical understanding and insight into the way mistuning affects the eigenvalues and eigenvectors of the system. The equations of motion for mistuned compressor blades are derived in the present paper. These equations make use of Lane's[7] mode shapes for the tuned system and they employ generalized unsteady aerodynamic coefficients. The novelty in these equations involves the use of a special form of interpolated aerodynamic coefficients as a function of the reduced frequency. Since Edwards' work[8] on unsteady aerodynamic modeling for arbitrary motions, Padé approximations for aerodynamic coefficients have been used extensively in aeroelastic problems involving active controls.[9,10] The success of this aerodynamic modeling has led to its increasing usage in other aeroelastic problems. In a very recent paper,[11] Padé approximants have also been introduced to the problem of cascade blade mistunings. However, a characteristic feature of the Padé representation involves the introduction of aerodynamic poles which lead to augmented states in the aeroelastic equations. In the present work, it is shown that the representation of the aerodynamic coefficients as complex quadratic functions of the reduced frequency $k$ (which avoids the introduction of aerodynamic poles) yields excellent results. This leads to simplified equations of motion which yield the important analytical result that the sum of the real parts of all the eigenvalues of a mistuned system (zero mean mistuning is sometimes required), is independent of the mistuning of the system.

The aerodynamic energy approach is used in this work with the object of gaining some physical insight into the mechanism through which mistuning affects flutter. However, this approach eventually yields in Part II the means required to identify the most stable flutter eigenvectors. The simplified equations developed herein will enable us, in Part II, to analytically determine the mistunings associated with these eigenvectors.

For simplicity, the equations are derived for blade systems involving torsional oscillations only. In Part II of this work it is shown that all blades with bending-torsion freedoms can be reduced to "equivalent" systems involving torsional oscillations only. All the numerical results presented in this work (both parts) are obtained by solving the "exact" equations of motion. The simplified equations of motion are used for analytical work only. The analytical results are then tested using exact equations of motion.

## Mathematical Model and Analytical Discussion

### Derivation of the Equations of Motion

The equation of motion of a single blade (with aerodynamic forces) is given by

$$[-\omega^2 I_{\alpha_s} + \omega_{\alpha_s}^2 (1 + ig_{\alpha_s}) I_{\alpha_s}]\alpha_s = M_s \tag{1}$$

where $s$ denotes the $s$th blade, $I_\alpha$ the rotational moment of inertia, $\omega_\alpha$ the natural undamped frequency of torsional oscillation, $\omega$ the frequency of torsional oscillation, $g_\alpha$ the structural damping coefficient in torsion, $\alpha$ the rotational amplitude of oscillation, and $M$ the aerodynamic moment.

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The uncoupled equations of motion for all the blades can be written as

$$\left[-\omega^2\begin{bmatrix} I_{\alpha 1} & & \\ & I_{\alpha 2} & \\ & & \ddots \\ & & & I_{\alpha N}\end{bmatrix} + \begin{bmatrix}\omega_{\alpha 1}^2(1+ig_{\alpha 1})I_{\alpha 1} & & \\ & \omega_{\alpha 2}^2(1+ig_{\alpha 2})I_{\alpha 2} & \\ & & \ddots \\ & & & \omega_{\alpha N}^2(1+ig_{\alpha N})I_{\alpha N}\end{bmatrix}\right]\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \vdots \\ \alpha_N\end{Bmatrix} = \begin{Bmatrix}M_1\\ M_2\\ \vdots\\ M_N\end{Bmatrix} \tag{2}$$

where $N$ denotes the number of blades. Lane had shown in his classical work[7] that for tuned compressor blades, the mode shapes of vibration are such that the amplitudes of oscillation of all the blades are the same and that each blade leads or lags its neighbors by a constant phase angle. Each mode shape has a different value of phase angle. Lane had shown that these phase angles, denoted by $\beta_r$, are given by

$$\beta_r = (2\pi/N)r; \qquad r = 0,1,\ldots,N-1 \tag{3}$$

Hence, the mode shapes matrix is given by

$$[E] = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{i\beta_1} & e^{i\beta_2} & \cdots & e^{i\beta_{N-1}} \\ 1 & e^{i2\beta_1} & e^{i2\beta_2} & \cdots & e^{i2\beta_{N-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{i(N-1)\beta_1} & e^{i(N-1)\beta_2} & \cdots & e^{i(N-1)\beta_{N-1}} \end{bmatrix} \tag{4}$$

Using the mode shapes derived by Lane and expressed by Eq. (4), Whitehead[12] derived the generalized aerodynamic forces for incompressible two-dimensional flow. For the case of torsional oscillation treated in this work, these generalized forces can be written by

$$\{M\} = \pi\rho U^2 C^2 [E] \begin{bmatrix} C_{M_\alpha}(0) & & & \\ & C_{M_\alpha}(\beta_1) & & \\ & & C_{M_\alpha}(\beta_2) & \\ & & & \ddots \\ & & & & C_{M_\alpha}(\beta_{N-1}) \end{bmatrix} \times \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_n \end{Bmatrix} \tag{5}$$

where $U$ is the airspeed, $C$ the chord of the blade, $\rho$ the air density, $C_{M_\alpha}(\beta_r)$ the generalized aerodynamic moment coefficient about the torsional elastic axis relating to the mode with $\beta_r$ as phase difference between adjacent blades, and $q_i$ is the coefficient of modal participation for the $i$th mode. However,

$$\begin{Bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{Bmatrix} = [E]\begin{Bmatrix} q_1\\ q_2\\ \vdots\\ q_n\end{Bmatrix} \tag{6}$$

or

$$\{\alpha\} = [E]\{q\} \tag{7}$$

and, therefore,

$$\{q\} = [E]\{\alpha\} \tag{8}$$

Substituting Eq. (8) in Eq. (5) one obtains

$$\{M\} = \pi\rho U^2 C^2 [E] \lceil C_{M\alpha} \rfloor [E]^{-1}\{\alpha\} \tag{9}$$

where $\lceil C_{M\alpha}\rfloor$ is a diagonal matrix defined by

$$\lceil C_{M\alpha}\rfloor = \begin{bmatrix} C_{M\alpha}(0) & & \\ & C_{M\alpha}(\beta_1) & \\ & & \ddots \\ & & & C_{M\alpha}(\beta_{N-1}) \end{bmatrix} \tag{10}$$

Introducing the aerodynamic forces given by Eq. (9) (which need not be limited to Whitehead's coefficients) into the equations of motion [Eq. (2)], the following equations result:

$$(-\omega^2 \lceil I_\alpha \rfloor + \lceil \omega_\alpha^2(1+ig_\alpha)I_\alpha \rfloor)\{\alpha\} = \pi\rho U^2 C^2 [E]\lceil C_{M\alpha}\rfloor [E]^{-1}\{\alpha\} \tag{11}$$

where

$$\lceil I_\alpha \rfloor = \begin{bmatrix} I_{\alpha 1} & & \\ & I_{\alpha 2} & \\ & & \ddots \\ & & & I_{\alpha N}\end{bmatrix} \tag{12}$$

and

$$\lceil \omega_\alpha^2(1+ig_\alpha)I_\alpha \rfloor = \begin{bmatrix}\omega_{\alpha 1}^2(1+ig_{\alpha 1})I_{\alpha 1} & & \\ & \omega_{\alpha 2}^2(1+ig_{\alpha 2})I_{\alpha 2} & \\ & & \ddots \\ & & & \omega_{\alpha N}^2(1+ig_{\alpha N})I_{\alpha N}\end{bmatrix} \tag{13}$$

For the case of tuned blades, all the diagonal terms in Eqs. (12) and (13) will be identical. For the mistuned case, the following notation will be used.

$$I_{\alpha s} = I_\alpha\left(1 + \frac{\Delta I_{\alpha s}}{100}\right) \tag{14}$$

$$\omega_{\alpha s}^2 I_{\alpha s} = \omega_\alpha^2 I_\alpha \left[1 + \frac{\Delta(\omega_{\alpha s}^2 I_{\alpha s})}{100}\right] \tag{15}$$

where $\omega_\alpha$ and $I_\alpha$ are the nominal (usually tuned) values from which mistuning is measured. Substituting Eqs. (14) and (15) into Eq. (11) the following equation is obtained:

$$(-\omega^2 I_\alpha \lceil \bar{I}_\alpha \rfloor + \omega_\alpha^2 I_\alpha \lceil \bar{K}_\alpha \rfloor)\{\alpha\} = \pi\rho U^2 C^2 [E] \lceil C_{M\alpha} \rfloor [E]^{-1}\{\alpha\} \tag{16}$$

where

$$\lceil \bar{I}_\alpha \rfloor = \begin{bmatrix} 1 + \frac{\Delta I_{\alpha 1}}{100} & & & \\ & 1 + \frac{\Delta I_{\alpha 2}}{100} & & \\ & & \ddots & \\ & & & 1 + \frac{\Delta I_{\alpha N}}{100} \end{bmatrix} \tag{17}$$

$$\lceil \bar{K}_\alpha \rfloor = \begin{bmatrix} \left[1 + \frac{\Delta(\omega_{\alpha 1}^2 I_{\alpha 1})}{100}\right][1 + ig_{\alpha 1}] & & & \\ & \left[1 + \frac{\Delta(\omega_{\alpha 2}^2 I_{\alpha 2})}{100}\right][1 + ig_{\alpha 2}] & & \\ & & \ddots & \\ & & & \left[1 + \frac{\Delta(\omega_{\alpha N}^2 I_{\alpha N})}{100}\right][1 + ig_{\alpha N}] \end{bmatrix} \tag{18}$$

Note that the matrices in Eqs. (17) and (18) are both diagonal and that $\Delta I_{\alpha s}$, $\Delta(\omega_{\alpha s}^2 I_{\alpha s})$ represent percentages of mistuning in moment of inertia and structural stiffness, respectively, as related to the $s$th blade.

Equation (16) can be nondimensionalized by dividing both sides by $\pi\rho b^2 \omega_\alpha^2 C^2$, where $b$ denotes the semichord length (i.e., $0.5C$), to obtain

$$\left(-\frac{\omega^2}{\omega_\alpha^2} M_R \bar{r}_\alpha^2 \lceil \bar{I}_\alpha \rfloor + M_R \bar{r}_\alpha^2 \lceil \bar{K}_\alpha \rfloor\right)\{\alpha\} = \frac{4U^2}{\omega_\alpha^2 C^2}[E] \lceil C_{M\alpha} \rfloor [E]^{-1}\{\alpha\} \tag{19}$$

where $\bar{r}_\alpha$ is the nondimensional radius of gyration defined by

$$\bar{r}_\alpha^2 = I_\alpha / mC^2 \tag{20}$$

and where $m$ is the mass per unit span of the blade. $M_R$ denotes the mass ratio defined by

$$M_R = m/\pi\rho b^2 \tag{21}$$

The matrix $[C_{M\alpha}]$ in Eq. (19) is a function of the reduced frequency $k_c (= \omega C/U)$. The normal approach of solving Eq. (19) starts with an assumed value of $k_c$ and solves for the eigenvalue $\omega$. It yields eigenvalues which correspond to different airspeeds. In order to ensure eigensolutions corresponding to the same speed, an alternative approach is adopted. It is assumed that over a range of values the aerodynamic coefficients are correct also for complex values of $k_c$. Such an approximation is widely used in flutter problems involving active controls and the aerodynamic coefficients are represented as a Padé approximation involving $k_c$ (Ref. 8). However, as already stated, the Padé approximation has the drawback that it leads to higher order differential equations of motion, and it, therefore, results in a larger order eigenvalue problem. To avoid an unnecessary increase in the order of the equations of motion, the following expression is used to represent the variation of the aerodynamic coefficients with $k_c$:

$$C_{M\alpha}(\beta_s) = C_0 + ik_c C_1 + (ik_c)^2 C_2 \tag{22}$$

As shown in the Appendix, the preceding approximation of $C_{M\alpha}(\beta_s)$ gives relatively poor results when $C_0$, $C_1$, $C_2$ are constrained to assume real values only. This is similar to the case in active controls, which leads to the introduction of the rational expressions in $ik_c$ (that is, the Padé approximation). However, when $C_0$, $C_1$, $C_2$ are permitted (in the present work) to assume complex values, excellent results for real values of $k_c$ are obtained by the approximation represented by Eq. (22). Hence, Eq. (22) with complex $C_j$ values is adopted in the present work. For further details regarding this approximation, see the Appendix and Table 1.

Introducing Eq. (22) into Eq. (19) and rearranging, the following equation is obtained.

$$(-\bar{\omega}^2[M_R \bar{r}_\alpha^2 \lceil \bar{I}_\alpha \rfloor - 4[E] \lceil C_2 \rfloor [E]^{-1}] - i\bar{\omega}[4\bar{U}[E] \lceil C_1 \rfloor [E]^{-1}] - [4\bar{U}^2[E] \lceil C_0 \rfloor [E]^{-1} - M_R \bar{r}_\alpha^2 \lceil \bar{K}_\alpha \rfloor])\{\alpha\} = 0 \tag{23}$$

where

$$\bar{\omega} = \omega/\omega_\alpha \tag{24}$$

$$\bar{U} = U/\omega_\alpha C \tag{25}$$

Equation (23) can be written briefly as

$$(-\bar{\omega}^2[A] - i\bar{\omega}[B] - [D])\{\alpha\} = 0 \tag{26}$$

where, by comparison with Eq. (23), it follows that

$$[A] = M_R \bar{r}_\alpha^2 \lceil \bar{I}_\alpha \rfloor - 4[E] \lceil C_2 \rfloor [E]^{-1} \tag{27}$$

$$[B] = 4\bar{U}[E] \lceil C_1 \rfloor [E]^{-1} \tag{28}$$

$$[D] = 4\bar{U}^2[E] \lceil C_0 \rfloor [E]^{-1} - M_R \bar{r}^2 \lceil K_a \rfloor \tag{29}$$

Once the nondimensional speed $\bar{U}$ is specified, the matrices $[A]$, $[B]$, and $[D]$ are completely determined and Eq. (26) can be reduced to a first-order equation following a well-known procedure, that is, let

$$\{X\} = \begin{Bmatrix} \dot{\alpha} \\ \alpha \end{Bmatrix} \tag{30}$$

where $\{X\}$ is a column of length $2N$, and $\dot{\alpha}$ is the derivative with respect to time of the vector $\alpha$. Since

$$\{\dot{\alpha}\} = i\bar{\omega}\{\alpha\} \tag{31}$$

Eqs. (26) and (31) can be brought to the following form:

$$i\bar{\omega}\begin{bmatrix} [A] & 0 \\ 0 & I \end{bmatrix}\begin{Bmatrix} \dot{\alpha} \\ \alpha \end{Bmatrix} + \begin{bmatrix} -[B] & -[D] \\ -I & 0 \end{bmatrix}\begin{Bmatrix} \dot{\alpha} \\ \alpha \end{Bmatrix} = 0 \tag{32}$$

or, after rearranging

$$\lambda\begin{Bmatrix} \dot{\alpha} \\ \alpha \end{Bmatrix} = \begin{bmatrix} [A]^{-1}[B] & [A]^{-1}[D] \\ I & 0 \end{bmatrix}\begin{Bmatrix} \dot{\alpha} \\ \alpha \end{Bmatrix} \tag{33}$$

where $\lambda = i\bar{\omega}$.

Equation (33) is a $2N \times 2N$ complex eigenvalue equation. Since the physical system involves $N$ degrees of freedom only, it follows that $N$ frequencies will relate to "nonphysical" solutions and will have to be discarded. It can be shown that these nonphysical eigenvalues are those which yield a negative value for the frequency, or, in other words, those eigenvalues that yield a negative imaginary part of $\lambda$.

### Coupling Through Mistuning

Consider Eq. (19) and assume that mistuning is introduced by means of changes in the stiffness terms along the diagonal of matrix $\lceil \bar{K}_\alpha \rfloor$. Using Eq. (7), and premultiplying Eq. (19) by $[E]^{-1}$, the following equation is obtained:

$$(-\bar{\omega}^2 M_R \bar{r}_\alpha^2 [E]^{-1} \lceil \bar{I}_\alpha \rfloor [E] + M_R \bar{r}_\alpha^2 [E]^{-1} \lceil \bar{K}_\alpha \rfloor [E])\{q\} = 4\bar{U}^2 \lceil C_{M\alpha} \rfloor \{q\} \tag{34}$$

When the system is tuned, all the diagonal terms in $\lceil \bar{K}_\alpha \rfloor$ are identical and equal to $(1+ig)$. Hence, for tuned system

$$[E]^{-1} \lceil \bar{K}_\alpha \rfloor [E] = (1+ig) \lceil I \rfloor \tag{35}$$

where $\lceil I \rfloor$ is the unit matrix. However, when the diagonal terms of $\lceil \bar{K} \rfloor$ are not identical, the product $[E]^{-1} \lceil \bar{K}_\alpha \rfloor [E]$ yields a matrix with nonzero off-diagonal terms. Similar considerations apply when mistuning is introduced into the $\lceil \bar{I}_\alpha \rfloor$ matrix. Therefore, mistuning in either stiffness or mass introduces coupling between modes.

## Sensitivity of Mode Shapes to Mistuning

The effect of mistuning essentially manifests itself through the modifications introduced in the mode shapes. It is, therefore, of prime importance to determine the sensitivity of the mode shapes of a compressor blade system to the amount of mistuning.

A compressor blade system is typified by high mass ratios and, therefore, the inertia and stiffness forces are much larger than the aerodynamic forces. In the example presented in later sections of this work, and which relates to the NASA test rotor 12,[3,4] the aerodynamic forces are two orders of magnitude smaller than the inertia and stiffness forces. It was noted before that Eq. (33) yields $2N$ different frequencies and eigenvectors, and that $N$ of these eigenvalues have negative imaginary parts and do not represent a physical solution. The remaining $N$ solutions, with positive imaginary parts in their eigenvalues, are the solutions which are of interest to us. Since the aerodynamic forces are small and $\lambda$ is nondimensionalized with respect to the tuned blade frequency $\omega_\alpha$, the imaginary parts of $\lambda$ will assume $N$ values around $+i$ and $N$ values around $-i$. To avoid the nonphysical solutions, Eq. (26) is transformed to a new set of eigenvalues $\bar{\lambda}$, where

$$\bar{\lambda} = \lambda - i \tag{36}$$

That is, Eq. (26) is replaced by

$$(\bar{\lambda}^{2}[A] - \bar{\lambda}([B] - 2i[A]) - [A] - [D] - i[B])\{\alpha\} = 0 \tag{37}$$

However, by virtue of Eq. (36), the values of $\bar{\lambda}$ associated with the physical eigenvalues will be very small compared with the values of $\bar{\lambda}$ associated with the nonphysical solutions (which will be of order $-2i$). Hence, ignoring $\bar{\lambda}^2 [A]$ in Eq. (37) yields

$$(-\bar{\lambda}([B] - 2i[A]) - [A] - [D] - i[B])\{\alpha\} = 0 \tag{38}$$

Therefore, Eq. (38) yields the $N$ physical eigenvalues and eigenvectors, or, alternatively,

$$\bar{\lambda}\{\alpha\} = (2i[A] - [B])^{-1} [\,[A] + [D] + i[B]\,]\{\alpha\} \tag{39}$$

Referring to Eqs. (27-29) and remembering that the aerodynamic forces are two orders of magnitude smaller than the inertia and stiffness forces, one can write the following approximation:

$$(2i[A] - [B])^{-1} \approx \frac{-i}{2M_R \bar{r}_\alpha^2} \left[ \frac{1}{\bar{I}_\alpha} \right] \tag{40}$$

where $[1/\bar{I}_\alpha]$ is a diagonal matrix which represents the inverse of $[\bar{I}_\alpha]$. However,

$$\begin{aligned}[A] + [D] + i[B] &= M_R \bar{r}_\alpha^2 (\lceil \bar{I}_\alpha \rfloor - \lceil \bar{K}_\alpha \rfloor) \\ &+ 4\bar{U}^2 [E] \lceil C_0 \rfloor [E]^{-1} - 4[E] \lceil C_2 \rfloor [E]^{-1} \\ &+ i4\bar{U}[E] \lceil C_1 \rfloor [E]^{-1}\end{aligned} \tag{41}$$

Hence,

$$\begin{aligned}\bar{\lambda}\{\alpha\} = \Bigg( \frac{i}{2} \left( \left[ \frac{1}{\bar{I}_\alpha} \right] \lceil \bar{K}_\alpha \rfloor - \lceil I \rfloor \right) \\ + \frac{1}{2M_R \bar{r}_\alpha^2} \left[ \frac{1}{\bar{I}_\alpha} \right] [4\bar{U}[E] \lceil C_1 \rfloor [E]^{-1} \\ - i4\bar{U}^2 [E] \lceil C_0 \rfloor [E]^{-1} + i4[E] \lceil C_2 \rfloor [E]^{-1}] \Bigg) \{\alpha\}\end{aligned} \tag{42}$$

For illustration purposes, assume that mistuning is introduced in the stiffness matrix only, i.e., in $\lceil \bar{K}_\alpha \rfloor$. In this case

$$\left[ \frac{1}{\bar{I}_\alpha} \right] = \lceil I \rfloor \tag{43}$$

and

$$\lceil \bar{K}_\alpha \rfloor - \lceil I \rfloor = \left[ \frac{\Delta\omega_\alpha^2 I_\alpha}{100} \right] (1+ig) \tag{44}$$

Equation (42) reduces in this case to

$$\begin{aligned}\bar{\lambda}\{\alpha\} = \frac{i}{2}(1+ig) \left[ \frac{\Delta\omega_\alpha^2 I_\alpha}{100} \right] + \frac{1}{2M_R \bar{r}_\alpha^2} [\,[E] (4\bar{U} \lceil C_1 \rfloor \\ - i4\bar{U}^2 \lceil C_0 \rfloor + i4 \lceil C_2 \rfloor)[E]^{-1}]\{\alpha\}\end{aligned} \tag{45}$$

Remembering that the expression involving the aerodynamic terms in Eq. (45) is of second order, it follows that if the mistuning terms are made to be of the first order (that is around 10% mistuning), then these mistuning terms dominate Eq. (45), and, as a result, they also dominate both the $\bar{\lambda}$ eigenvalues and the $\bar{\lambda}$ and $\lambda$ eigenvectors. In practical cases, mistuning of smaller order than 10% may be sufficient to ensure the necessary changes in the eigenvectors. In later sections of this work, use will be made of Eqs. (42) an (45) while attempting to assess different forms of mistuning.

## Effect of Mistuning on the Real Part of $\lambda$

The mistuning considered in this work will be limited to a mistuning with zero mean value, that is,

$$\sum_{s=1}^{N} \Delta I_{\alpha s} = 0 \tag{46}$$

and

$$\sum_{s=1}^{N} \Delta(\omega_{\alpha_s}^2 I_{\alpha_s}) = 0 \tag{47}$$

since nonzero mean values of mistuning are equivalent to a change in design of the tuned system which is not within the scope of the present work.

It has already been shown that mistuning affects the coupling between the different $q$ modes. It would, therefore, be interesting to study how this coupling effect manifests itself on the real part of the eigenvalues (which expresses the exponential rate of decay, or rate of increase, of the various oscillations).

Denote

$$\lambda = \mu + i\nu \tag{48}$$

then, from Eq. (36),

$$\bar{\lambda} = \mu + i(\nu - 1) \tag{49}$$

The effects of mistuning on $\mu$ can most conveniently be studied using Eq. (42). Assume that mistuning is introduced in the stiffness terms only. Equations (45) and (47) indicate that the sum of all the $\mu$'s of the eigenvalues must remain constant, since the trace of the lambda matrix remains unaffected by the introduction of mistuning.

Hence,

$$\sum_{s=1}^{N} \mu_s = \text{const} \tag{50}$$

Note that if no structural damping exists (i.e., $g_\alpha = 0$), Eq. (50) is satisfied even if Eq. (47) is not satisfied. This is true since, in this case, the expression involving the $\bar{K}_\alpha$ terms is purely imaginary and, therefore, has no effect on Eq. (50) which involves real terms only. This result should come as no surprise since it can be readily verified that $\mu$ is independent of the stiffness in a single degree of freedom system with no structural damping. If mistuning is introduced in the mass matrix, Eq. (50) is still satisfied provided Eq. (46) is also satisfied and provided the amount of mistuning is not excessively large. This limitation on the amount of mistuning follows the approximation

$$1 \Big/ \left(1 + \frac{\Delta I_\alpha}{100}\right) \approx 1 - \frac{\Delta I_\alpha}{100} \tag{51}$$

which is introduced in the $[1/\bar{I}_\alpha]$ diagonal matrix. However, since it has already been shown (in earlier sections of this work) that the amount of necessary mistuning is small, it follows that Eq. (51) represents a very satisfactory approximation (mass mistuning is further discussed in a following section of this work).

Equation (50) represents an important result, which is in line with previous discussions presented in this work, i.e.: Any improvement in $\mu_j$ caused by means of mistuning must be accompanied by the deterioration of one or more values $\mu_i$ associated with other modes.[2,3] Hence, there exists an upper bound for the stabilizing effects of mistuning. This upper bound is reached when all the different $\mu$'s are equal, and, hence, by virtue of Eq. (50), their value is given by

$$\mu_i = \sum_{s=1/N}^{N} \mu_s = \mu_{\text{mean}}; \quad i = 1,2,\ldots,N \tag{52}$$

Hence, by means of Eqs. (50) and (52) the maximum stabilizing effect of mistuning can be determined, even without the determination of the form of mistuning.

## Effects of Mass Mistunings

The results presented so far relate to mistuned stiffness terms. A question, therefore, arises regarding the effects of mistuned mass terms. Equation (42) shows that there must be a difference between the results obtained by stiffness mistunings and those obtained by mass mistunings. One of these theoretical differences [see Eqs. (50) and (51)] was discussed earlier in this work. It will, however, be shown that for practical purposes, when considering the values involved, the differences obtained between the results due to mistunings in stiffness and those due to mistunings in mass are negligible.

It has already been shown that the amounts of mistunings involved are of the first order. Hence, the approximation represented by Eq. (51) can be substituted into Eq. (42) to yield

$$\begin{aligned}\bar{\lambda}\{\alpha\} = \Bigg(&\frac{i}{2}[(\lceil I \rfloor - \lceil \Delta I_\alpha \rfloor)\lceil \bar{K}_\alpha \rfloor - \lceil I \rfloor] \\ &+ \frac{1}{2M_R \bar{r}_\alpha^2}(\lceil I \rfloor - \lceil \Delta I_\alpha \rfloor)[4\bar{U}[E]\lceil C_1 \rfloor [E]^{-1} \\ &- i4\bar{U}^2[E]\lceil C_0 \rfloor [E]^{-1} + i4[E]\lceil C_2 \rfloor [E]^{-1}]\Bigg)\{\alpha\}\end{aligned} \tag{53}$$

Substitution of Eq. (18), with zero structural damping (and with zero mistunings in stiffness), into Eq. (53) yields

$$\begin{aligned}\lambda\{\alpha\} = \Bigg(&-\frac{i}{2}\left[\frac{\Delta I_\alpha}{100}\right] + \frac{([I] - [\Delta I_\alpha])}{2M_R \bar{r}_\alpha^2}[4\bar{U}[E][C_1][E]^{-1} \\ &- i4\bar{U}[E][C_0][E]^{-1} + i4[E][C_2][E]^{-1}]\Bigg)\{\alpha\}\end{aligned} \tag{54}$$

The terms along the diagonal of the aerodynamic matrices are all identical (for aerodynamically tuned blades), and, therefore, the requirement for zero trace of $[\Delta I_\alpha]$ is sufficient to insure $\mu_{\text{mean}} = \text{const}$. It can be seen that Eq. (50) is satisfied for mass mistunings only if the mistunings are small, the trace of $[\Delta I_\alpha]$ is zero, and the blades are aerodynamically tuned. It should be recalled that stiffness mistunings, with no structural damping, impose no such restrictions on either the magnitudes of the mistunings, or on the trace of $[\Delta\omega_\alpha^2 I_\alpha]$.

## Blade Mistuning Using the Aerodynamic Energy Approach

To gain some physical insight, the problem of mistuned blades with no structural damping (i.e., $g_\alpha = 0$) will be studied using the aerodynamic energy approach.[13] It can be shown[13] that the work done by a system [such as the one represented by Eq. (20)] on its surroundings during one cycle of hamonic oscillation is given by $P$, where

$$P = \frac{\pi^2 \rho U^2 C^2}{2} \lfloor \alpha^* \rfloor [W]\{\alpha\} \tag{55}$$

where $[W]$ is a Hermitian matrix defined by

$$[W] = -([\bar{A}_I] + [\bar{A}_I]^T) + i([\bar{A}_R] - [\bar{A}_R]^T) \tag{56}$$

where $\bar{A}_R$ is the real part of $([E] \lceil C_{M\alpha} \rfloor [E]^{-1})$ and $\bar{A}_I$ the imaginary part of $([E] \lceil C_{M\alpha} \rfloor [E]^{-1})$; and where $[\ ]^T$ denotes the transpose matrix and an asterisk the complex conjugate. Hence, $\bar{A}_R$ and $\bar{A}_I$ can be written as

$$\bar{A}_R = ½([E]\lceil C_{M\alpha} \rfloor [E]^{-1} + [E]^* \lceil C_{M\alpha} \rfloor^* [E^{-1}]^* \tag{57a}$$

$$\bar{A}_I = ½([E]\lceil C_{M\alpha} \rfloor [E]^{-1} - [E]^* \lceil C_{M\alpha} \rfloor^* [E^{-1}]^* \tag{57b}$$

It can be readily shown[7] that

$$[E]^* = N[E]^{-1} \tag{58}$$

and, therefore,

$$[E^{-1}]^* = 1/N[E] \tag{59}$$

Substitution of Eqs. (58) and (59) into Eqs. (57a) and (57b) yields

$$\bar{A}_R = \tfrac{1}{2}([E]\lceil C_{M\alpha}\rfloor [E]^{-1} + [E]^{-1}\lceil C^*_{M\alpha}\rfloor [E]) \qquad (60)$$

$$\bar{A}_I = \tfrac{1}{2}([E]\lceil C_{M\alpha}\rfloor [E]^{-1} - [E]^{-1}\lceil C^*_{M\alpha}\rfloor [E]) \qquad (61)$$

Noting that $\lceil C_{M\alpha}\rfloor$ is a diagonal matrix and that $[E]$ and $[E]^{-1}$ are symmetrical matrices, $[W]$ can be shown to be

$$[W] = -\frac{1}{2i}[[E]\lceil C_{M\alpha}\rfloor [E]^{-1} - [E]^{-1}\lceil C^*_{M\alpha}\rfloor [E] + [E]^{-1}\lceil C_{M\alpha}\rfloor [E] - [E]\lceil C^*_{M\alpha}\rfloor [E]^{-1}] + \tfrac{1}{2}[[E]\lceil C_{M\alpha}\rfloor [E]^{-1} + [E]^{-1}\lceil C^*_{M\alpha}\rfloor [E] - [E]^{-1}\lceil C_{M\alpha}\rfloor [E] - [E]\lceil C^*_{M\alpha}\rfloor [E]^{-1}]$$

or

$$[W] = i[E](\lceil C_{M\alpha}\rfloor - [C^*_{M\alpha}])[E]^{-1} \qquad (62)$$

or

$$[W] = 2[E]\lceil -C_{M\alpha_I}\rfloor [E]^{-1} \qquad (63)$$

where

$$\lceil C_{M\alpha_I}\rfloor = \text{imaginary part of } \lceil C_{M\alpha}\rfloor \qquad (64)$$

Hence, using Eqs. (55) and (64)

$$P = \pi^2\rho U^2 C^2 \lfloor\alpha^*\rfloor [E]\lceil -C_{M\alpha_I}\rfloor [E]^{-1}\{\alpha\} \qquad (65)$$

$$\{\alpha\} = [E]\{q\} \qquad (66)$$

where $q$ represents a vibration mode. Hence, $\{\alpha^*\} = [E^*]\{q^*\}$. Using Eq. (58) $\{\alpha^*\}$ can be written as

$$\{\alpha^*\} = N[E]^{-1}\{q^*\} \qquad (67)$$

Substitution of Eqs. (66) and (67) into Eq. (65) yields

$$P = \pi^2\rho U^2 C^2 N\lfloor q^*\rfloor \lceil -C_{M\alpha_I}\rfloor \{q\} \qquad (68)$$

Equation (68) can be expanded to yield the following alternative form.

$$P = \pi^2\rho U^2 C^2 N[(q_{1R}^2 + q_{1I}^2)(-C_{M\alpha_1})_I + (q_{2R}^2 + q_{2I}^2)(-C_{M\alpha_2})_I + \cdots + (q_{NR}^2 + q_{NI}^2)(-C_{M\alpha_N})_I] \qquad (69)$$

## Discussion of the Aerodynamic Energy Results

Equation (69) has the following properties.

1) The work done by the system on its surroundings is not directly dependent on the inertia and stiffness terms which influence the work only through their influence on the modes.

2) For a dissipative system, $P$ must be positive.

3) $P$ will always be positive if all $(C_{M\alpha_j})_I$ are negative. If a single value of $(C_{M\alpha_j})_I$ is positive, a mode shape exists ($q_i = 0$ for $i = 1,2,\ldots,$ except for $i = j$) which drives the system unstable. The generalized $q_i$ coordinates represent the normal modes of oscillation of the tuned system.[1] It, therefore, follows from Eq. (69) that a tuned blade system will be unstable if, and only if, at least one value of $(C_{M\alpha_j})_I > 0$. Assume that such an instability of the tuned system exists, with only one value of $(C_{M\alpha_j})_I > 0$. Any coupling introduced in the system which will excite some or all of the other modes, in addition to the $j$th unstable mode, is shown by Eq. (68) to be accompanied by some dissipation and, therefore, will always be stabilizing. Hence, the tuned blade system is the least stable system. This is always the case when the physical mode shapes coincide with the aerodynamic energy mode shapes (see also Ref. 13). Furthermore, the stabilization of the $j$th mode by means of coupling must be accompanied by a reduction in stability of the other modes. Assume that coupling is introduced between the $j$th (unstable) mode and the $i$th (stable) one. Since $q_j$ will (by virtue of the coupling) always be accompanied by $q_i$, this coupling will be dissipative and, therefore, stabilizing. However, when mode $i$ is disturbed, $q_i$ (by virtue of the coupling) will always be accompanied by $q_j$, which reduces the total dissipation and is, therefore, destabilizing. Hence, it can be seen that the stabilization of a tuned blade system by means of coupling (i.e., mistuning) will always be accompanied by some reduction in stability of other modes.

The preceding discussion can be formulated mathematically as follows. Define the specific work $\bar{P}$ as the work done when the Euclidean norm of the $\alpha$'s is equal to 1, that is,

$$\bar{P} = \frac{\lfloor\alpha^*\rfloor [\bar{W}]\{\alpha\}}{\lfloor\alpha^*\rfloor \{\alpha\}} \qquad (70)$$

where $[\bar{W}]$ is Hermitian and given by [see Eq. (68)]

$$[\bar{W}] = \pi^2\rho U^2 C^2 [E]\lceil -C_{M\alpha_I}\rfloor [E]^{-1} \qquad (71)$$

However, since $\bar{P}$ is expressed as a Rayleigh quotient, its maximum and minimum values correspond to the maximum and minimum eigenvalues of $[\bar{W}]$. Furthermore, $[\bar{W}]$ is similar to the matrix $\pi^2\rho U^2 C^2[-C_{M\alpha I}]$, and so they both have the same eigenvalues. Hence,

$$\bar{P}_{\max} = -\pi^2\rho U^2 C^2 (C_{M\alpha_I})_{\min}$$
$$\bar{P}_{\min} = -\pi^2\rho U^2 C^2 (C_{M\alpha_I})_{\max} \qquad (72)$$

From Eqs. (66) and (68), it is clear that $\bar{P}_{\max}$ and $\bar{P}_{\min}$ are attained for the eigenvectors of $[\bar{W}]$ which are the columns of $[E]$, or the modes of the tuned system. Hence, it can be seen once again, that the tuned system yields the least stable and the most stable modes. The mistuned system will yield modes with stabilities bounded by the extreme stabilities of the tuned system. These results agree with those of Whitehead[1,2] (see also Refs. 3 and 4), who showed that mistuning is beneficial for flutter but may lead to increased responses in other modes (although Ref. 5 indicates that Whitehead's results are disputed by a Russian paper).

## The Existence of Optimum Coupling

Assume that coupling is introduced between the different $q_i$ coordinates, so that the new mode shapes of vibrations are a linear combination of Lane's[7] tuned system mode shapes. These new mode shapes will be referred to as the "$q$ coupled modes." Let us further assume that after the introduction of coupling, one of the $q$ coupled modes dissipates less energy per cycle than another $q$ coupled mode. It follows from the aerodynamic energy analysis and the discussion that followed it, that the dissipation of the lightly dissipative $q$ coupled mode can be further increased if additional coupling is introduced so as to couple these two $q$ coupled modes. A similar argument can be repeated until all the $q$ coupled modes yield the same amount of energy dissipation per cycle of oscillation (based on some norm of the $q$ responses). It can thus be seen that although the aerodynamic energy approach is unable to yield the optimum $q$ coupled mode, it may be helpful in assessing any eigenvector obtained in relation to the aforementioned most stable eigenvector.

**Table 1 Comparison between exact values of $C_{F\alpha}(\beta_{II})$ and interpolated values, together with percentage errors. Complex interpolation matrices, 12 blades, and 10 value interpolation ($1.15 \le k_c \le 2.35$)**

| $k_c$ | Interpolated value of $\mathrm{Re}(C_{F\alpha})$ | Exact value of $\mathrm{Re}(C_{F\alpha})$ | Percentage error in $\mathrm{Re}(C_{F\alpha})$, % | Interpolated value of $\mathrm{Im}(C_{F\alpha})$ | Exact value of $\mathrm{Im}(C_{F\alpha})$ | Percentage error in $\mathrm{Im}(C_{F\alpha})$, % |
|---|---|---|---|---|---|---|
| 1.15 | −0.253989D 00 | −0.254200D 00 | 0.488083D-01 | −0.350877D 00 | −0.350300D 00 | 0.133410D 00 |
| 1.30 | −0.223522D 00 | −0.223400D 00 | 0.251897D-01 | −0.431063D 00 | −0.431300D 00 | 0.487262D-01 |
| 1.40 | −0.200743D 00 | −0.200600D 00 | 0.271702D-01 | −0.484296D 00 | −0.484700D 00 | 0.770529D-01 |
| 1.50 | −0.175988D 00 | −0.175900D 00 | 0.154819D-01 | −0.537348D 00 | −0.537700D 00 | 0.621409D-01 |
| 1.60 | −0.149258D 00 | −0.149200D 00 | 0.944792D-02 | −0.590221D 00 | −0.590400D 00 | 0.293536D-01 |
| 1.75 | −0.105459D 00 | −0.105500D 00 | 0.603589D-02 | −0.669193D 00 | −0.669100D 00 | 0.137680D-01 |
| 1.90 | −0.572167D-01 | −0.573600D-01 | 0.191184D-01 | −0.747761D 00 | −0.747400D 00 | 0.481110D-01 |
| 2.05 | −0.453026D-02 | −0.465000D-02 | 0.145054D-01 | −0.825923D 00 | −0.825500D 00 | 0.512876D-01 |
| 2.20 | 0.526002D-01 | 0.525800D-01 | 0.223009D-02 | −0.903681D 00 | −0.903500D 00 | 0.200545D-01 |
| 2.35 | 0.114175D 00 | 0.114300D 00 | 0.126877D-01 | −0.981035D 00 | −0.981500D 00 | 0.470604D-01 |

**Table 2 Variation of the generalized moment coefficient $C_{M\alpha}(\beta_s)$ with reduced frequency $k_c$, $N=12$**

| S | $C_{M\alpha}(\beta_s)$ at $k_c=2.05$ | | $C_{M\alpha}(\beta_s)$ at $k_c=1.9$ | | $C_{M\alpha}(\beta_s)$ at $k_c=1.75$ | | $C_{M\alpha}(\beta_s)$ at $k_c=1.5$ | |
|---|---|---|---|---|---|---|---|---|
| | Re | Im | Re | Im | Re | Im | Re | Im |
| 0 | 0.141290D 00 | −0.753890D-01 | 0.135893D 00 | −0.698875D-01 | 0.131069D 00 | −0.643787D-01 | 0.123825D 00 | −0.551200D-01 |
| 1 | 0.123420D 00 | −0.186035D-01 | 0.122423D 00 | −0.128170D-01 | 0.122061D 00 | −0.681000D-02 | 0.122943D 00 | 0.386750D-02 |
| 2 | 0.138417D 00 | −0.723750D-03 | 0.139934D 00 | 0.496250D-02 | 0.142185D 00 | 0.107500D-01 | 0.147935D 00 | 0.210750D-01 |
| 3 | 0.171311D 00 | −0.581750D-02 | 0.174048D 00 | −0.940000D-03 | 0.177703D 00 | 0.417500D-02 | 0.186038D 00 | 0.133750D-01 |
| 4 | 0.209971D 00 | −0.317750D-01 | 0.212803D 00 | −0.279050D-01 | 0.216531D 00 | −0.236500D-01 | 0.224838D 00 | −0.164250D-01 |
| 5 | 0.244970D 00 | −0.727950D-01 | 0.246368D 00 | −0.696150D-01 | 0.249019D 00 | −0.665500D-01 | 0.255578D 00 | −0.615500D-01 |
| 6 | 0.268619D 00 | −0.120487D 00 | 0.267430D 00 | −0.118805D 00 | 0.267700D 00 | −0.116888D 00 | 0.270556D 00 | −0.113950D 00 |
| 7 | 0.275433D 00 | −0.166738D 00 | 0.271393D 00 | −0.165020D 00 | 0.269100D 00 | −0.164063D 00 | 0.266881D 00 | −0.162175D 00 |
| 8 | 0.262876D 00 | −0.199618D 00 | 0.256353D 00 | −0.198208D 00 | 0.250838D 00 | −0.197250D 00 | 0.243774D 00 | −0.195888D 00 |
| 9 | 0.232556D 00 | −0.210598D 00 | 0.224237D 00 | −0.208190D 00 | 0.216421D 00 | −0.207031D 00 | 0.205489D 00 | −0.205413D 00 |
| 10 | 0.191430D 00 | −0.192510D 00 | 0.182015D 00 | −0.189525D 00 | 0.173020D 00 | −0.187050D 00 | 0.159360D 00 | −0.184162D 00 |
| 11 | 0.151904D 00 | −0.144817D 00 | 0.142938D 00 | −0.140452D 00 | 0.134219D 00 | −0.136451D 00 | 0.120235D 00 | −0.130472D 00 |

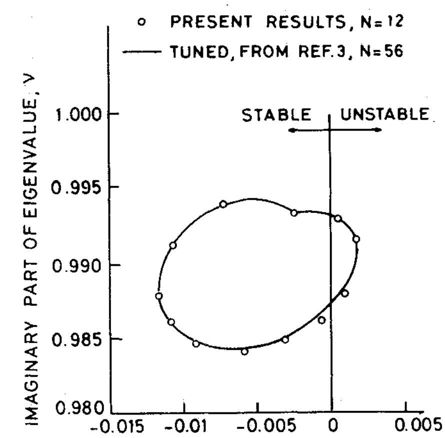


**Fig. 1 Comparison between present results and those from Ref. 3 (with $N=56$) for the tuned system, $g=0$, $U=0.787$.**

## Results and Discussion

### Description of the Numerical Example

All the results presented herein relate to the NASA test rotor 12 which is modified herein to include 12 blades instead of its 56 blades.[3,4] The elastic axis position and the c.g. position are assumed to be at the midchord point, unless otherwise stated. Additional data relate to the stagger angle $\xi = 54.4$ deg, the radius of gyration (normalized with respect to the chord) $\bar{r}_\alpha = 0.2887$, the mass ratio $M_R = 258.5$, and the gap $S$ between blades (normalized with respect to the chord) $S/C = 0.534$. The reduction in the number of blades is introduced in order to reduce the computational labor while testing the different aspects associated with mistuning. In addition, the blades are allowed torsional freedom only (instead of the bending torsion freedom allowed in Ref. 4). For elastic axis position at midchord, with c.g. also at midchord, the elimination of the bending degree of freedom is known to have negligible effects.[3] The case where the elastic axis does not coincide with the c.g. position will be treated in Part II of this work.

### Results for the Tuned Blades

The present formulation of the equations of motion, based on the interpolated aerodynamic matrices, yields all eigenvalues at the same speed. This is in contrast to the methods which assume a value for the reduced frequency $k$ and then proceed to determine the airspeed which is generally different for each eigenvalue. For this reason, and by virtue of the difference in the number of blades, a comparison between the tuned blades results obtained herein and those given in Ref. 3 is appropriate. Figure 1 shows a very good agreement between the two sets of results. The nondimensional flutter speed obtained herein for the tuned system is given by $\bar{U}_F \approx 0.5$. Since the flutter speed in Ref. 3 is normalized with respect to the semichord b, and in this work the chord C is used to normalize all the lengths (except for the mass ratio $M_R$), the flutter speeds differ by a factor of 2. After allowing for the normalization effect, the flutter speed reported in Ref. 3 is $\bar{U}_F = 0.48$. Here, again, the comparison is satisfactory.

Table 2 shows the variation of $C_{M_{\alpha_s}}$ with $k_c$. It can be seen that at $k_c = 1.9$ the system is unstable since there exists a $(C_{M_{\alpha_j}})_I > 0$. Furthermore, the critical flutter mode is associated with $\beta_2 = 60$ deg, that is, when successive blades lead their predecessors by 60 deg. At lower reduced frequencies (such as when $k_c = 1.75$ and $k_c = 1.5$) the 30-deg phase lead and the 90-deg phase lead modes also turn to yield positive $(C_{M_\alpha})_I$ values, and, therefore, lead to instability. However, in all cases, the largest value of positive $(C_{M_\alpha})_I$ is associated with the 60-deg phase lead and it is therefore concluded that this is the least stable mode.

**Table 3 Eigenvalues relating to purely alternate stiffness mistuning with values ±6% at $\bar{U}=0.5$ and $\bar{U}=0.787$, $N=12$**

| Purely alternate stiffness mistunings, % | Eigenvalue at $\bar{U}=0.5$ Re | Im | Eigenvalue at $\bar{U}=0.787$ Re | Im |
|---|---|---|---|---|
| −6 | −0.232017D-02 | 0.964653D+00 | −0.543482D-02 | 0.101935D+01 |
| 6 | −0.226480D-02 | 0.964807D+00 | −0.581751D-02 | 0.101847D+01 |
| −6 | −0.245670D-02 | 0.964838D+00 | −0.514757D-02 | 0.101779D+01 |
| 6 | −0.250982D-02 | 0.964836D+00 | −0.348117D-02 | 0.101838D+01 |
| −6 | −0.260594D-02 | 0.964850D+00 | −0.411641D-02 | 0.101785D+01 |
| 6 | −0.258577D-02 | 0.964844D+00 | −0.419830D-02 | 0.101877D+01 |
| −6 | −0.200126D-02 | 0.102507D+01 | −0.492982D-02 | 0.957429D+00 |
| 6 | −0.254741D-02 | 0.102498D+01 | −0.503625D-02 | 0.957875D+00 |
| −6 | −0.248724D-02 | 0.102509D+01 | −0.504207D-02 | 0.958118D+00 |
| 6 | −0.239482D-02 | 0.102492D+01 | −0.518840D-02 | 0.958377D+00 |
| −6 | −0.216497D-02 | 0.102498D+01 | −0.537838D-02 | 0.958505D+00 |
| 6 | −0.219868D-02 | 0.102497D+01 | −0.540220D-02 | 0.958319D+00 |

### Results for Alternate Mistuning

Table 3 presents the eigenvalues obtained [by solving Eq. (19); see also, Eq. (33)] when stiffness mistuning is introduced in the numerical example described earlier (with $\bar{U}=0.5$). The stiffness mistuning used has purely alternate values of ±6%, which are within the first-order values discussed earlier in this work. It can be seen that smaller values are obtained for $\mu_{max}$, when compared with those obtained for the tuned system. Furthermore, the small $\mu$ values increase so that $\mu_{mean}$ remains constant, as predicted earlier in this work. In the following, the eigenvectors relating to the aforementioned purely alternate mistuned system will be studied in an attempt to gain some physical insight into the nature of this solution.

Table 4 presents a typical eigenvector obtained from the aforementioned purely alternate mistuned system. These results indicate that each alternate blade oscillates with almost the same amplitude, but with a phase difference of $2\beta_r$ [see Eq. (3)], whereas the intermediate blades remain relatively stationary (that is, between one and two orders of magnitude smaller in amplitude relative to their two adjoining blades). From Eqs. (58) and (66), it follows that $\{q\}$ is obtained from

$$\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_4 \end{Bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-i\beta_1} & e^{-i\beta_2} & \cdots & e^{-i\beta_{N-1}} \\ 1 & e^{-i2\beta_1} & e^{-i2\beta_2} & \cdots & e^{-i2\beta_{N-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{-i(N-1)\beta_2} & e^{-i(N-1)\beta_2} & & e^{-i(N-1)\beta_{N-1}} \end{bmatrix} \times \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \\ \alpha_N \end{Bmatrix} \tag{73}$$

Hence, the purely alternate mistuning yields a set of $\{q\}$ vectors (after ignoring the high orders of $\alpha$'s relative to dominant $\alpha_j$), consisting of two nonzero equal terms in each vector, that is, $q_j$ and $q_{j+N/2}$, $j=1,2,\ldots,N/2$. These two terms appear once in phase and once in antiphase, thus leading to a total of $N$ eigenvectors. Since purely alternate mistuning requires an even number of blades, $N/2$ is always an integer.

At this stage, one may ask why such vectors, as obtained for the purely alternate mistuning, are as effective as they are from the point of view of dissipation of energy. A glance at Table 2 ($k_c=1.9$, which is around flutter) shows that the coupling between $q_j$ and $q_{j+(N/2)}$ (as obtained from the purely alternate mistuning), couples $(-C_{M_{\alpha_I}})_j$ and $(-C_{M_{\alpha_I}})_{j+(N/2)}$ which has the property that when $(-C_{M_{\alpha_I}})_j$ is smallest $(-C_{M_{\alpha_I}})_{j+(N/2)}$ is largest. Note that Table 2 has values of $(N-1)\geq j\geq 0$, instead of $N/2\geq j\geq 1$ appearing in Table 4. When $(-C_{M_{\alpha_I}})_j$ is slightly larger than the minimum, $(-C_{M_{\alpha_I}})_{j+(N/2)}$ is slightly smaller than the maximum. Since the modulus of each of the $q_j$'s appearing in Table 4 has the same identical value, it follows from Eq. (69) that the dissipation of energy in this case will depend on the sum of $(-C_{M_{\alpha_I}})_j$ and $(-C_{M_{\alpha_I}})_{j+(N/2)}$. These sums vary between 0.1778 and 0.21743, thus indicating that some of the responses dissipate more energy than the others, so that further coupling will be beneficial.

### Physical Significance of the Results

The nature of the compressor blade flutter instability originates from the aerodynamic interference on a specific blade caused by the motion of its neighboring blades. This motion is such that a specific blade may either lead its predecessor or lag behind it depending on the value of $\beta_r$ [Eq. (3)]. About half of the mode shapes of the tuned system will show successive blades leading their predecessors, and about half of the mode shapes will show successive blades lagging their predecessors [see Eqs. (3), (54), and (66)]. It is the existence of this very large number of phase angles ($=N$), which span all four quadrants, that lead to the flutter instability. In very simple terms, it may be stated that the damping of a single blade due to its own motion depends on the normal velocity of its ¾ chord point. Hence, for a single blade, zero aerodynamic damping will be obtained only when the elastic axis of a one degree of freedom torsional blade coincides with this ¾ chord point (excluding the Glauert instability, which occurs at extremely low vlaues of reduced frequency). In all other cases, the normal velocity sensed by this ¾ chord point will be in a direction which is opposite to its rotational velocity, thus leading, always, to positive aerodynamic damping coefficients.

In the case treated in this work, the flow induced at this ¾ chord point of a blade is greatly affected by the movements of its neighboring blades. These neighboring blades produce, by their own motion, vortices which span the length of the blades and their wakes up to infinity. Hence, as a result of this flow, which is induced by the neighboring blades, the ¾ chord point will not always sense an air velocity in a direction opposite to its rotational velocity, but may even sense an air velocity in the same direction as its rotational velocity. Such cases lead to negative aerodynamic dampings and (as a result) to compressor blade flutter. As stated earlier, the phase angles of the oscillations of the neighboring blades span the four quadrants in intervals of $2\pi/N$. Hence, there will always be a phase angle (for $N$ sufficiently large) that will result in a destabilizing induced flow.

**Table 4 Typical eigenvector of α's relating to the purely alternate stiffness mistunings of ±6%, at $\bar{U}=0.5$, $N=12$**

| Purely alternate stiffness mistunings, % | α eigenvector at $\bar{U}=0.5$ Re | Im | Modulus of α $\lvert\alpha\rvert$ |
|---|---|---|---|
| −6 | −0.2673D+00 | −0.2047D-01 | 0.26808D 00 |
| 6 | 0.6931D−02 | −0.1994D-02 | 0.72121D-02 |
| −6 | −0.2673D+00 | −0.2047D-01 | 0.26808D 00 |
| 6 | 0.6931D−02 | −0.1994D-02 | 0.72121D-02 |
| −6 | −0.2673D+00 | −0.2047D-01 | 0.26808D 00 |
| 6 | 0.6931D−02 | −0.1994D-02 | 0.72121D-02 |
| −6 | −0.2673D+00 | −0.2047D-01 | 0.26808D 00 |
| 6 | 0.6931D−02 | −0.1994D-02 | 0.72121D-02 |
| −6 | −0.2673D+00 | −0.2047D-01 | 0.26808D 00 |
| 6 | 0.6931D−02 | −0.1994D-02 | 0.72121D-02 |
| −6 | −0.2673D+00 | −0.2047D-01 | 0.26808D 00 |
| 6 | 0.6931D−02 | −0.1994D-02 | 0.72121D-02 |

The purely alternate mistuning yields results whereby every other blade oscillates with a phase difference (between these two blades) of $2\beta_r$ while the interim blades are stationary. This leads to reduced interference effects due to two reasons. 1) The two blades adjoining a given oscillating blade are stationary and, as a result, their effects on the induced velocity is greatly reduced. It is these two blades that contribute most to the induced velocity due to their close proximity to the oscillating blade. 2) The number of phase angles associated with the moving blades is reduced by half, thus creating less opportunities to hit a critical phase angle. The foregoing discussion will be extended in Part II of this work to yield the optimum mode shapes for maximum stability.

## Concluding Remarks

The formulation of the equations of motion with interpolated aerodynamic coefficients, using a complex quadratic variation with reduced frequency, leads to relatively low-order equations. These equations are found to be amenable to analytical treatment and yield the important result regarding the invariance with mistuning of the sum of the real parts of the eigenvalues. This leads to the conclusion that the upper bound for stabilization through mistuning is reached when all real parts of the eigenvalues are equal to the invariant mean of their sum. Furthermore, any stabilization of a mode through mistuning must be accompanied by a deterioration of stability for another mode. The aerodynamic energy approach yields some insight into the problem of stabilization through mistuning and it indicates that optimum coupling through mistuning is obtained when all coupled modes dissipate equal amounts of energy per cycle of oscillation.

## Appendix: Interpolation of the Aerodynamic Matrices

The aerodynamic forces acting on a blade that oscillates with a phase difference of $\beta_r$ relative to its neighboring blades, and which is a part of a cascade of blades, can be shown[12] to be given by

$$F=\pi\rho UC(\dot{h}C_{F\dot{h}}+\alpha UC_{F\alpha}-WC_{Fw}) \quad \text{(A1)}$$

$$M=\pi\rho UC^2(\dot{h}C_{M\dot{h}}+\alpha UC_{M\alpha}-WC_{Mw}) \quad \text{(A2)}$$

where $\dot{h}$ denotes the translational velocity of the leading edge (positive downward), $\alpha$ the rotation about the leading edge (positive clockwise), $w$ the velocity of disturbance due to wakes from upstream obstructions, $F$ the lift force (positive downward), and $M$ the moment about the leading edge (positive clockwise). The coefficients $C_{F\dot{h}}$, $C_{M\dot{h}}$, $C_{F\alpha}$, $C_{M\alpha}$, $C_{Fw}$, and $C_{Mw}$ are all nondimensional, and are functions of the reduced frequency $k_c(=\omega C/U)$, $\beta_r$ (for incompressible flow), the stagger angle $\xi$, and the cascade spacing parameter $s/c$. Equations (A1) and (A2) can be brought to the following form so as to avoid an indirect dependence on the frequency of oscillation $\omega$ (through $\dot{h}$):

$$F=\pi\rho U^2C\left(ik_cC_{F\dot{h}}\frac{h}{c}+C_{F\alpha}\alpha-C_{Fw}\frac{w}{U}\right) \quad \text{(A3)}$$

$$M=\pi\rho U^2C^2\left(ik_cC_{M\dot{h}}\frac{h}{c}+C_{M\alpha}\alpha-C_{Mw}\frac{w}{U}\right) \quad \text{(A4)}$$

An attempt will now be made to express explicitly the dependence of the aforementioned coefficients on $k_c$ by using the following representation:

$$[C]=[A_0]+ik_c[A_1]+(ik_c)^2[A_2] \quad \text{(A5)}$$

where $[C]$ is given by

$$[C]=\begin{bmatrix} ik_cC_{F\dot{h}} & C_{F\alpha} & C_{FW} \\ ik_cC_{M\dot{h}} & C_{M\alpha} & C_{Mw} \end{bmatrix} \quad \text{(A6)}$$

and the matrices $[A_0]$, $[A_1]$, and $[A_2]$ are all of order $(2\times 3)$.

Assume that a set of $n_k$ matrices $[C]$ [Eq. (A6)] exists, with each matrix referring to a different value of $k_c$. The $(i,j)$ coefficients of the different $A$ matrices [Eq. (A5)] can be found by solving the following equations, using least square techniques.

$$[R_k]=\begin{Bmatrix} a_{0i,j} \\ a_{1i,j} \\ a_{2i,j} \end{Bmatrix}=\begin{Bmatrix} T_c \end{Bmatrix}_{i,j} \quad \text{(A7)}$$

If one assumes that all the $[A]$ matrices are real, then in this case, $[R_k]$ is given by

$$[R_k]=\begin{bmatrix} 1 & 0 & -k_{c_1}^2 \\ 0 & k_{c_1} & 0 \\ 1 & 0 & -k_{c_2}^2 \\ 0 & k_{c_2} & 0 \\ \vdots & & \\ 1 & 0 & -k_{c_{nk}}^2 \\ 0 & k_{c_{nk}} & 0 \end{bmatrix} \quad \text{(A8)}$$

and $\{T_c\}_{i,j}$ is given by

$$\{T_c\}_{i,j} = \begin{Bmatrix} \mathrm{Re}(C_{ij})_{k=k_1} \\ \mathrm{Im}(C_{ij})_{k=k_1} \\ \mathrm{Rm}(C_{ij})_{k=k_2} \\ \mathrm{Im}(C_{ij})_{k=k_2} \\ \mathrm{Re}(C_{ij})_{k=k_{nk}} \\ \mathrm{Im}(C_{ij})_{k=k_{nk}} \end{Bmatrix} \tag{A9}$$

If, however, one assumes that all the $[A]$ matrices are complex, then in this case, $[R_k]$ is given by

$$[R_k] = \begin{bmatrix} 1 & ik_1 & -k_1^2 \\ 1 & ik_2 & -k_2^2 \\ \vdots & & \\ 1 & ik_{nk} & -k_{nk}^2 \end{bmatrix} \tag{A10}$$

and $\{T_c\}_{i,j}$ is given by

$$\{T_c\}_{i,j} = \begin{Bmatrix} (C_{ij})_{k=k_1} \\ (C_{ij})_{k=k_2} \\ \vdots \\ (C_{ij})_{k=k_{nk}} \end{Bmatrix} \tag{A11}$$

In all cases, the coefficients represented by $[a_{0ij}\ a_{1ij}\ a_{2ij}]^T$ are obtained from Eq. (A7) by means of least squares, i.e.,

$$\begin{Bmatrix} a_{0ij} \\ a_{1ij} \\ a_{2ij} \end{Bmatrix} = ([R_k]^T[R_k])^{-1}[R_k]^T\{T_c\}_{i,j} \tag{A12}$$

Ten values of $k_c$ were used, ranging from $k_c = 1.15$ to $k_c = 2.35$. These values of $k_c$ were chosen to cover the flutter reduced frequencies of a numerical example used herein. The quality of the results obtained is judged by the differences between the exact values of any $C_{ij}$, for the different values of $k_c$, and the corresponding ones obtained from Eq. (A5). The results obtained when constraining the interpolation matrices to real values only showed large errors of up to 60%.

Table 1 presents typical results obtained when the $[A]$ matrices are permitted to assume complex values. In all cases the errors in Table 1 are much smaller than 1%, with error values of around 1% in very rare occasions (for other $i,j$ values not shown). Hence, it is concluded that the representation of the aerodynamic matrices by a quadratic polynomial in $(ik)$, with complex matrix coefficients, yields excellent results, and there is no need to improve the approximation by the introduction of Padé approximants (which increase the number of states, leading to increased order systems). This result is very interesting and can be applied to a variety of aeroelastic problems, including aeroelasticity with active controls.


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